# Functional Analysis

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Lecture 11

## Hahn-Banach theorem

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**Lem1.** Let $X$ be a complex normed space. A functional $f : X \to \mathbb{C}$ is $\mathbb{C}$-linear $\iff f(x) = u(x) + iu(-ix)$, where $u : X \to \mathbb{R}$ is $\mathbb{R}$-linear. Moreover, $\|f\| = \|u\|$.

**Proof:** "$\Longrightarrow$" If $f$ is $\mathbb{C}$-linear, then $u := \operatorname{Re} f$ is $\mathbb{R}$-linear and

$$\operatorname{Im} f(x) = \operatorname{Im}[i(-i)f(x)] = \operatorname{Im} if(-ix) = \operatorname{Re} f(-ix) = u(-ix).$$

Hence $f(x) = u(x) + iu(-ix)$. Moreover,

$$\|u\| = \sup_{\|x\|=1} |u(x)| = \sup_{\|x\|=1} |\operatorname{Re} f(x)| \leqslant \sup_{\|x\|=1} |f(x)| = \|f\|.$$

"$\Longleftarrow$" If $u : X \to \mathbb{R}$ is $\mathbb{R}$-linear, then $f(x) := u(x) + iu(-ix)$ defines a functional $f : X \to \mathbb{C}$, which is $\mathbb{C}$-linear.

Let $x \in X$, $\|x\| = 1$. Take $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda f(x) = |f(x)|$. Then

$$|f(x)| = \lambda f(x) = f(\lambda x) = \operatorname{Re} f(\lambda x) = u(\lambda x) \leqslant \|u\|.$$

That is $\|f\| \leqslant \|u\|$. ■

**Def.** Let $X$ be a linear space over $\mathbb{R}$. A **Banach functional** is a function $p : X \to \mathbb{R}$ such that

1. $\forall_{x,y \in X}\ p(x+y) \leqslant p(x) + p(y)$, (*triangle inequality*)
2. $\forall_{x \in X}\ \forall_{t>0}\ p(tx) = tp(x)$. (*positive homogenity*)

**Ex.** Linear functionals, norms and semi-norms.

**Thm. (Banach lemma)**

$$\left( \begin{array}{l} X_0 \subseteq X \text{ linear subspace} \\ f_0 : X_0 \to \mathbb{R} \text{ linear functional} \\ p : X \to \mathbb{R} \text{ Banach functional} \\ \forall_{x \in X_0}\ \ f_0(x) \leqslant p(x) \end{array} \right) \Longrightarrow \left( \begin{array}{l} \text{there exists a linear} \\ \text{functional } f : X \to \mathbb{R} \\ \text{such that } f|_{X_0} = f_0 \\ \forall_{x \in X}\ \ f(x) \leqslant p(x) \end{array} \right)$$

**Proof:** We prove it in two steps:

1) We apply Kuratowski-Zorn lemma, to obtain a maximal extension $f$ of $f_0$, which is majorized by $p$.
2) We show that this maximal extension is defined on whole $X$.

1) Let $\Phi$ be the set of all linear extensions dominated by $p$, i.e.

$$\Phi := \{(\varphi, X_\varphi) : X_\varphi \text{ linear subspace containing } X_0, \quad \varphi : X_\varphi \to \mathbb{R} \text{ linear functional} \quad \text{such that } \varphi|_{X_0} = f_0 \text{ and } \varphi \leqslant p|_{X_\varphi}\}.$$

We introduce the partial order relation on $\Phi$ as follows:

$$(\varphi_1, X_{\varphi_1}) \prec (\varphi_2, X_{\varphi_2}) \stackrel{def}{\iff} X_{\varphi_1} \subseteq X_{\varphi_2} \text{ oraz } \varphi_2|_{X_{\varphi_1}} = \varphi_1.$$

Every linearly ordered set$\{(\varphi_i, X_i)\}_{i\in I}$ has an upper bound $(\varphi, X_\varphi)$, where $X_\varphi := \bigcup_{i\in I} X_{\varphi_i}$ and $\varphi(x) := \varphi_i(x)$ if $x \in X_{\varphi_i}$. Therefore, by the Kuratowski–Zorn Lemma, there exists a maximal element $(\varphi_m, X_{\varphi_m})$ in $\Phi$.

2) If $X_{\varphi_m} = X$, then by setting $f = \varphi_m$ we finish the proof. Let us assume indirectly that there exists $x_0 \in X \setminus X_{\varphi_m}$ and let us put

$$\widetilde{X} := \text{lin}\{X_{\varphi_m}, x_0\} = \{y + \lambda x_0 : x \in X_{\varphi_m},\ \lambda \in \mathbb{R}\}.$$

For any $u \in \mathbb{R}$ the formula

$$\varphi_u(x + \lambda x_0) := \varphi_m(x) + \lambda u, \qquad x \in X_{\varphi_m},\ \lambda \in \mathbb{R},$$

defines a linear functional $\varphi_u : \widetilde{X} \to \mathbb{R}$, which extends $\varphi_m$. 

**Question**: Can we pick $u \in \mathbb{R}$ so that $p$ dominates $\varphi_1$?

$$\underset{\widetilde{x}\in\widetilde{X}}{\forall}\ \varphi_u(\widetilde{x}) \leqslant p(\widetilde{x}) \iff \underset{x\in X_{\varphi_m},\,\lambda\in\mathbb{R}}{\forall}\ \varphi_m(x) + \lambda u \leqslant p(x + \lambda x_0)$$

$$\iff \underset{x\in X_{\varphi_m},\,\lambda>0}{\forall} \begin{cases} u \leqslant \frac{1}{\lambda}\left(p(x+\lambda x_0) - \varphi_m(x)\right) \\ u \geqslant -\frac{1}{\lambda}\left(p(x-\lambda x_0) - \varphi_m(x)\right) \end{cases}$$

$$\overset{\text{homogen. } p}{\iff} \underset{x\in X_{\varphi_m},\,\lambda>0}{\forall} \begin{cases} u \leqslant p\left(\frac{x}{\lambda} + x_0\right) - \varphi_m\left(\frac{x}{\lambda}\right) \\ u \geqslant -p\left(\frac{x}{\lambda} - x_0\right) + \varphi_m\left(\frac{x}{\lambda}\right) \end{cases}$$

$$\iff \begin{cases} u \leqslant b := \inf\{p(x + x_0) - \varphi_m(x) : x \in X_{\varphi_m}\} \\ u \geqslant a := \sup\{-p(x - x_0) + \varphi_m(x) : x \in X_{\varphi_m}\}. \end{cases}$$

If $a \leqslant b$, then for any $u \in [a, b]$ we get $(\varphi_u, \widetilde{X}) \in \Phi$, which is an extension of maximal element $(\varphi_m, X_m) \in \Phi$ $\lightning$

For any $x_1, x_2 \in X_{\varphi_m}$ we have

$$
\begin{aligned}
p(x_1 + x_0) - \varphi_m(x_1) - \big( - p(x_2 - x_0) + \varphi_m(x_2)\big) \\
&= p(x_1 + x_0) + p(x_2 - x_0) - \varphi_m(x_1 + x_2) \quad \text{(triangle ineq. for } p\text{)} \\
&\geqslant p(x_1 + x_2) - \varphi_m(x_1 + x_2) \geqslant 0 \quad \big(\text{because } \varphi_m \leqslant p\big).
\end{aligned}
$$

Whence $b \geqslant a$. ∎

**Linear functionals extend in a norm preserving way !**

### Hahn-Banach Theorem

Let $X$ be a normed space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Every bounded linear functional $f_0 : X_0 \to \mathbb{F}$ defined on a subspace $X_0 \subseteq X$ **extends** to a bounded linear functional $f : X \to \mathbb{F}$ such that $\|f\| = \|f_0\|$.

**Proof:** (1) Assume that $\mathbb{F} = \mathbb{R}$. Then $p(x) := \|x\| \cdot \|f_0\|$, $x \in X$ is a Banach functional such that $f_0 \leqslant p$ on $X_0$. Therefore, by the Banach Lemma, there exists a linear functional $f : X \to \mathbb{R}$ such that $f|_{X_0} = f_0$ and $f \leqslant p$ on $X$. Hence

$$\|f\| = \sup_{\|x\|=1} |f(x)| \leqslant \sup_{\|x\|=1} p(x) = \sup_{\|x\|=1} \|x\| \cdot \|f_0\| = \|f_0\|.$$

Hence $f$ is bounded and $\|f\| \leqslant \|f_0\|$. For the reverse inequality:

$$\|f_0\| = \sup_{\substack{\|x\|=1 \\ x \in X_0}} |f_0(x)| = \sup_{\substack{\|x\|=1 \\ x \in X_0}} |f(x)| \leqslant \sup_{\substack{\|x\|=1 \\ x \in X}} |f(x)| = \|f\|.$$

(2) Assume that $\mathbb{F} = \mathbb{C}$. By **Lem1** $f_0(x) = u_0(x) + iu_0(-ix)$, where $u_0 : X_0 \to \mathbb{R}$ is $\mathbb{R}$-linear and $\|u_0\| = \|f_0\|$. From step (1) we know that $u_0$ extends to an $\mathbb{R}$-linear $u : X \to \mathbb{R}$ such that $\|u\| = \|u_0\| = \|f_0\|$. Hence again by **Lem1** we get that

$$f(x) := u(x) + iu(-ix), \qquad x \in X,$$

defines a $\mathbb{C}$-linear extension of $f_0$ and $\|f\| = \|u\| = \|f_0\|$. ■

**Cor1.** For each $x \in X \setminus \{0\}$ there is a functional $f \in X^*$ such that

$$\|f\| = 1 \qquad \text{oraz} \qquad f(x) = \|x\|.$$

In particular, bounded functionals **seperate points** in $X$, that is

$$\forall_{x,y \in X} \quad x \neq y \implies \exists_{f \in X^*} \quad f(x) \neq f(y).$$

**Proof:** Let $x \in X \setminus \{0\}$. Put $X_0 := \mathrm{lin}\{x\}$ and define $f_0 : X_0 \to \mathbb{F}$ by $f_0(\lambda x) := \lambda \|x\|$, $\lambda \in \mathbb{F}$. Then $f_0 \in X_0^*$ and $\|f_0\| = 1$. So $f_0$ extends to the desired functional $f$ by Hahn–Banach Thm.

In particular, if $x \neq y$, then $x - y \neq 0$, and so there is $f \in X^*$ such that $f(x-y) = \|x-y\| \neq 0$, whence $f(x) \neq f(y)$. ■

**Cor2.** Every normed space $X$ embeds into its **double dual** space

$$X^{**} := (X^*)^*.$$

Namely, we have a linear isometry $i : X \to X^{**}$ given by

$$i(x)(f) := f(x) \qquad x \in X, f \in X^*.$$

**Proof:** Let $x \in X$. The functional $i(x) : X^* \to \mathbb{F}$ is linear:

$$i(x)(\lambda f_1 + f_2) = (\lambda f_1 + f_2)(x) = \lambda f_1(x) + f_2(x) = \lambda i(x) f_1 + i(x) f_2$$

and $\|i(x)\|_{X^{**}} = \sup_{\|f\|_{X^*}=1} |i(x)(f)| = \sup_{\|f\|_{X^*}=1} |f(x)| \leqslant \|x\|_X$.

Hence $i(x) \in X^{**}$ and $\|i(x)\|_{X^{**}} \leqslant \|x\|_X$.

To prove the opposite inequality, we may assume that $x \neq 0$. Then by **Cor1** there is $f \in X^*$ such that $\|f\|_{X^*} = 1$ and $f(x) = \|x\|_X$. Hence $\|i(x)\|_{X^{**}} = \|x\|_X$. Thus the map

$$X \ni x \to i(x) \in X^{**}$$

is a well defined isometry. This isometry is linear beacuse

$$\begin{aligned} i(\lambda x_1 + x_2)(f) &= f(\lambda x_1 + x_2) = \lambda f(x_1) + f(x_2) \\ &= \lambda i(x_1)(f) + i(x_2)(f) = \big(\lambda i(x_1) + i(x_2)\big)(f). \quad \blacksquare \end{aligned}$$

**Def.** $X$ is a **reflexive space** if $i : X \to X^{**}$ is an isomorphism, that is if every bounded functional on $X^*$ is of the form $X^* \ni f \mapsto f(x) \in \mathbb{F}$ for some $x \in X$.

## Examples

| Reflexive | Non-reflexive |
|---|---|
| finite dimensional spaces,<br>Hilbert spaces,<br>$L^p$-spaces for $1 < p < \infty$ | $c_0$, $C[a,b]$, $\ell^1$, $\ell^\infty$,<br>$L^1([a,b])$, $L^\infty([a,b])$ |

## Thm. (Duals to $L^p$-spaces)

For any $1 \leqslant p < \infty$ and measure space $(\Omega, \Sigma, \mu)$ ($\sigma$-finite when $p = 1$) we have a natural isometric isomorphism

$$L^p(\mu)^* \cong L^q(\mu), \qquad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

More precisely, for any $f \in L^p(\mu)^*$ there is $y \in L^q(\mu)$ such that

$$f(x) = \int_\Omega x \cdot y \, d\mu, \qquad x \in L^p(\mu).$$

Moreover, then $\|f\| = \|y\|_q = \begin{cases} \left(\int_\Omega |y|^q \, d\mu\right)^{\frac{1}{q}}, & p > 1, \\ \text{ess sup}|y|, & p = 1. \end{cases}$

**Cor1.** $(\ell^p)^* \cong \ell^q$ for $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Namely

$$f \in (\ell^p)^* \iff \exists!_{y \in \ell^q} \ \forall_{x \in \ell^p} \ f(x) = \sum_{k=1}^{\infty} x(k) y(k)$$

and then $\|f\| = \|y\|_q$.

**Cor2.** The spaces $L^p(\mu)$, $\ell^p$ for $1 < p < \infty$ are reflexive.

**Proof:** Since $p > 1$, there is finite $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ $(q = \frac{1}{p-1})$. Applying **Thm** twice

$$L^p(\mu)^{**} = (L^p(\mu)^*)^* \cong L^q(\mu)^* \cong L^p(\mu). \qquad \blacksquare$$

**Lem.** $c_0^* \cong \ell^1$. Namely $f \in c_0^* \iff$ there is $y \in \ell^1$ such that $f(x) = \sum_{k=1}^{\infty} x(k)y(k)$, for $x \in c_0$, and $\|f\| = \|y\|_1 = \sum_{k=1}^{\infty} |y(k)|$.

**Proof:** 

**Cor.** $c_0$ is not refelexive.

**Proof:** We have $c_0^{**} = (c_0^*)^* \cong (\ell^1)^* \cong \ell^\infty$, but $c_0 \ncong \ell^\infty$, because for instance $c_0$ is separable, while $\ell^\infty$ is not. $\blacksquare$